

L^2 -Betti numbers of totally disconnected groups and their approximation by Betti numbers of lattices

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ABSTRACT. The main result is a general approximation theorem for normalised Betti numbers for Farber sequences of lattices in totally disconnected groups. Further, we contribute some computations and complements to the general theory of L^2 -Betti numbers of totally disconnected groups.

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1. Introduction

The study of asymptotics of Betti numbers of sequences of lattices with increasing covolume in locally compact groups has a long history. In this article, we prove a general convergence result that holds for any Farber chain in any unimodular, totally disconnected group. Moreover, we show that the limit can be identified in a natural way with an L^2 -invariant associated with the locally compact group. L^2 -Betti numbers of certain unimodular, locally compact groups, in particular of automorphism groups of buildings associated to BN-pairs, were first considered by Dymara [11] and Davis-Dymara-Januszkiewicz-Okun [9]. However, note that it was conjectured already by Gaboriau [15] that the L^2 -invariants of a unimodular graph

might be invariants of its automorphism group. In his PhD thesis [22] the first named author developed a homological theory of L^2 -Betti numbers that applies to all unimodular, locally compact groups and generalizes the one by Lück [20, Chapter 6] for the discrete case, see also [23, 24]. Computations reduce to the discrete case whenever the locally compact group G possesses a lattice Γ as in this case the relation between the L^2 -Betti numbers is

$$(1.1) \quad \beta_k^{(2)}(G, \nu) = \frac{\beta_k^{(2)}(\Gamma)}{\text{covol}_\nu(\Gamma)}.$$

This result is proved by Kyed-Petersen-Vaes [17], the difficult case being non-uniform lattices. Ultimately, it relied on Gaboriau's deep work on L^2 -invariants for equivalence relations [14]. We present a short proof of Equation (1.1) for arbitrary lattices in totally disconnected groups in Section 5.4. This also gives a direct proof of proportionality of L^2 -Betti numbers of lattices in the same locally compact, totally disconnected group.

A fundamental result in the theory of L^2 -Betti numbers is Lück's approximation theorem [18] which expresses L^2 -Betti numbers as limits of normalised Betti numbers along a residual chain of finite index normal subgroups. Farber relaxed the condition of normality for the sequence of finite index subgroups in the approximation theorem to what is now called a *Farber sequence* [13]. The relevance of this notion was explained in the context of invariant random subgroups by Bergeron and Gaboriau in [5]. The notion of Farber sequence admits a probabilistic generalisation to the situation where the discrete group is replaced by a locally compact totally disconnected group and the chain of finite index subgroups by a sequence of lattices.

Throughout the entire article, let G denote a locally compact totally disconnected, second countable, unimodular group with Haar measure ν unless explicitly stated otherwise. We will denote by $\mathcal{K}(G)$ the ordered set of compact-open subgroups of G .

DEFINITION 1.1. *A sequence $(\Gamma_i)_{i \in \mathbb{N}}$ of lattices in G is a Farber sequence if for every compact open subgroup $K < G$ and for every right coset C of K the probability that a conjugate $g\Gamma_i g^{-1}$ meets $C \setminus \{e\}$ tends to zero as $i \rightarrow \infty$.*

Here the probabilities are taken with respect to the probability Haar measures on G/Γ_i . Note that one could replace *for every compact open subgroup $K < G$* by *for some compact open subgroup $K < G$* in the above definition.

A lattice $\Gamma < G$ is called cocompact if G/Γ is compact. Note that if Γ is cocompact, then there is some compact open subgroup $K < G$ so that every conjugate of Γ meets K only in $\{e\}$. The following is a family version of cocompactness.

DEFINITION 1.2. A family F of discrete subgroups of G is called uniformly discrete if there is a compact open subgroup $K < G$ so that every conjugate of a subgroup in F meets K only in $\{e\}$.

Next we turn to our main result (proved in Subsection 4.4) which provides an approximation theorem for Farber sequences of lattices in G . A G -CW-complex is a CW-complex with a G -action that permutes open cells such that for an open cell e with $ge = e$ the map $x \mapsto gx$ for $x \in e$, is the identity (cf. [35, II.1]). A contractible G -CW-complex whose stabiliser groups are compact and open is called a *topological model of G* . A topological model for G always exists and is unique up to G -homotopy equivalence [19]. We call a G -CW-complex cocompact if it consists of only finitely many G -cells.

THEOREM 1.3. Assume that G admits a topological model whose $(n+1)$ -skeleton is cocompact. Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a Farber sequence of lattices. Then

$$\beta_n^{(2)}(G, \nu) \leq \liminf_{i \rightarrow \infty} \frac{\beta_n(\Gamma_i)}{\text{covol}_\nu(\Gamma_i)} \quad \text{for all } i \in \{0, \dots, n\}.$$

If, in addition, $(\Gamma_i)_{i \in \mathbb{N}}$ is uniformly discrete, then

$$\beta_n^{(2)}(G, \nu) = \lim_{i \rightarrow \infty} \frac{\beta_n(\Gamma_i)}{\text{covol}_\nu(\Gamma_i)} \quad \text{for all } i \in \{0, \dots, n\}.$$

Note that the inequality in Theorem 1.3 is the *non-trivial* and more interesting inequality, when one follows the proof of Lück's approximation theorem in the discrete case. Indeed, it is usually the other inequality – sometimes called Kazhdan's inequality – which follows more easily. However, in our situation we were not able to prove Kazhdan's inequality without further assumptions.

Next we relate the theorem to the subject of invariant random subgroups which emerged in the last several years.

The Chabauty topology (cf. Appendix A) on the set Sub_G of closed subgroups of G turns Sub_G into a compact, metrisable space. The group G acts continuously on Sub_G by conjugation. The space of invariant random subgroups IRS_G of G is the space of G -invariant Borel probability measures on Sub_G endowed with the topology of weak convergence. Point measures δ_N of closed normal subgroups $N < G$ yield examples of invariant random subgroups. Another source of invariant random subgroups are lattices $\Gamma < G$. Indeed, the push forward of the normalised Haar measure on G/Γ under the measurable map

$$G/\Gamma \rightarrow \text{Sub}_G, \quad g\Gamma \mapsto g\Gamma g^{-1}$$

is an invariant random subgroup, denoted by $\mu_\Gamma \in \text{Sub}_G$. Statements about random conjugates of a lattice Γ always refer to the measure μ_Γ .

The relationship of convergence of invariant random subgroups and Farber chains is made precise in the following proposition.

PROPOSITION 1.4 (cf. Appendix A). *If $(\Gamma_i)_{i \in \mathbb{N}}$ is a Farber sequence then the sequence $(\Gamma_i)_{i \in \mathbb{N}}$ converges to the trivial subgroup seen as an invariant random subgroup. The converse is true provided the sequence of lattices is uniformly discrete.*

A pioneering study of invariant random subgroups in simple Lie groups has been done in [1, 2] by Abért, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet. They prove structural results on the space IRS_G for G being a simple Lie group with property (T) and a convergence result for uniformly discrete sequences of lattices in such G , similar to Theorem 1.3. We cannot expect interesting structural results on IRS_G in the generality of arbitrary totally disconnected groups. However, for totally disconnected groups of algebraic origin this was achieved by Gelander and Levit [16].

THEOREM 1.5 (Gelander-Levit). *Let G be the k -points of a simply connected simple linear algebraic group over a non-Archimedean local field k . Assume that G has property (T). Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a sequence of lattices whose covolumes tend to ∞ as $i \rightarrow \infty$. If the characteristic of k is positive, we additionally assume that $(\Gamma_i)_{i \in \mathbb{N}}$ is uniformly discrete. Then, we have that $(\Gamma_i)_{i \in \mathbb{N}}$ is a Farber sequence.*

Thus, our results in combination of the work of Gelander and Levit nicely complements the results in [2], which are more analytic in spirit and only apply to Lie groups.

We discuss applications to some specific examples of groups in Section 5 and finish the paper with some miscellaneous observations, including a discussion of the Connes embedding problem for Hecke-von Neumann algebra which is a byproduct of our efforts to prove Theorem 1.3.

2. Algebraic and homological aspects of Hecke algebras

2.1. The Hecke algebras of G and of a Hecke pair (G, K) . Let $K < G$ be a compact open subgroup. The pair (G, K) will be called a *Hecke pair*. We normalise the Haar measure to $\nu(K) = 1$.

The *Hecke algebra* $\mathcal{H}(G)$ of G is the convolution algebra of compactly supported, locally constant, and complex-valued functions on G . It is endowed with an involution given by $f^*(g) = \overline{f(g^{-1})}$. The *Hecke algebra* $\mathcal{H}(G, K)$ of the Hecke pair (G, K) is the convolution algebra of compactly supported, K -bi-invariant, and complex-valued functions on G . Note that

$$\mathcal{H}(G, K) = 1_K \mathcal{H}(G) 1_K = \bigoplus_{KgK \in K \backslash G / K} \mathbb{C} \cdot 1_{KgK},$$

where 1_{KgK} denotes the characteristic function of the double coset $KgK \in K \backslash G / K$. Since $K \subset G$ is open, $K \backslash G$ is discrete and by our normalisation the push-forward of the Haar measure is equal to the counting measure. Hence, the space of square integrable left K -invariant functions on G is naturally identified with $\ell^2(K \backslash G)$. We denote by 1_{Kg} the characteristic function of the set Kg . The set $\{1_{Kg} \mid Kg \in K \backslash G\}$ is an orthonormal basis of $\ell^2(K \backslash G)$. Convolution determines a natural representation

$$\pi: \mathcal{H}(G, K) \rightarrow B(\ell^2(K \backslash G))$$

of $\mathcal{H}(G; K)$ as bounded operators on $\ell^2(K \backslash G)$. In the sequel let $[P] \in \{0, 1\}$ denote the truth value of a mathematical expression P . We have

$$\begin{aligned} \pi(1_{KgK})(1_{Kh})(s) &= (1_{KgK} * 1_{Kh})(s) = \int_G [t \in KgK] \cdot [t^{-1}s \in Kh] \, d\nu(t) \\ &= \int_G [t \in KgK] \cdot [s^{-1}t \in h^{-1}K] \, d\nu(t) \\ &= \int_G [t \in KgK] \cdot [t \in sh^{-1}K] \, d\nu(t) \\ &= \int_G [t \in KgK \cap sh^{-1}K] \, d\nu(t) \\ &= \nu(KgK \cap sh^{-1}K). \end{aligned}$$

Thus, setting $a_{g,h}^s := \nu(KgK \cap sh^{-1}K)$, we obtain

$$1_{KgK} * 1_{Kh} = \sum_{Ks \in K \backslash G} a_{g,h}^s \cdot 1_{Ks}.$$

The numbers $a_{g,h}^s$ are either 0 or 1 since $KgK \cap sh^{-1}K$ is either empty or equal $sh^{-1}K$. Since K is compact, there are, for fixed $g, h \in G$, only finitely many $Ks \in K \backslash G$ such that $a_{g,h}^s \neq 0$.

Similarly to the computation above, we see that

$$(2.1) \quad 1_{KgK} * 1_{KhK} = \sum_{KsK \in K \backslash G / K} b_{g,h}^s \cdot 1_{KsK}$$

with

$$b_{g,h}^s := \nu(KgK \cap sKh^{-1}K).$$

The numbers $b_{g,h}^s$ are non-negative integers since $KgK \cap sKh^{-1}K$ is a finite union of right cosets of K . They are called the *structure constants* of the Hecke algebra $\mathcal{H}(G, K)$.

DEFINITION 2.1. *The linear functional*

$$\mathrm{tr}: \mathcal{H}(G, K) \rightarrow \mathbb{C}, \quad \mathrm{tr}\left(\sum_{KsK \in K \backslash G / K} a_{KsK} \cdot 1_{KsK}\right) := a_K.$$

defines a trace on the Hecke algebra $\mathcal{H}(G, K)$.

Indeed, the trace property follows from unimodularity:

$$\mathrm{tr}(1_{KgK} * 1_{KhK}) = \nu(KgK \cap Kh^{-1}K) = \nu(KhK \cap Kg^{-1}K) = \mathrm{tr}(1_{KhK} * 1_{KgK})$$

It is easy to see that tr is positive, unital and self-adjoint with respect to the natural involution $1_{KgK}^* = 1_{Kg^{-1}K}$. We conclude that $\mathcal{H}(G, K)$ is a complex $*$ -algebra with a unital, positive and faithful trace. Moreover, we obtain

$$\mathrm{tr}(T) = \langle \pi(T)(1_K), 1_K \rangle \quad \text{for } T \in \mathcal{H}(G, K).$$

2.2. The integral Hecke algebra. It is important for our approach that homological invariants of the Hecke pair (G, K) are defined not only in terms of its associated Hecke algebra, but in terms of some integral version of it – a canonical \mathbb{Z} -algebra sitting inside $\mathcal{H}(G, K)$ that plays the role of the integral group ring sitting inside the complex group.

We define the *integral Hecke algebra* $\mathbb{Z}[K \backslash G / K]$ to be the \mathbb{Z} -linear span of

$$\{1_{KgK} \mid KgK \in K \backslash G / K\} \subset \mathcal{H}(G, K).$$

By (2.1), $\mathbb{Z}[K \backslash G / K]$ is closed under multiplication provided $\nu(K) = 1$. For brevity, we will denote the integral Hecke algebra also by $\mathbb{Z}[G, K]$. The following lemma sheds some light on how to think of the integral Hecke algebra.

LEMMA 2.2. *The action of $\mathbb{Z}[G, K]$ on $\mathbb{Z}[K \backslash G]$ by convolution yields an identification of the ring $\mathbb{Z}[K \backslash G / K]$ with the ring*

$$\mathrm{hom}_{\mathbb{Z}}(\mathbb{Z}[K \backslash G], \mathbb{Z}[K \backslash G])^G$$

of right G -equivariant endomorphisms of $\mathbb{Z}[K \backslash G]$.

PROOF. It is clear that there is a natural homomorphism

$$\varphi: \mathbb{Z}[K \backslash G / K] \rightarrow \mathrm{hom}_{\mathbb{Z}}(\mathbb{Z}[K \backslash G], \mathbb{Z}[K \backslash G])^G,$$

which is given by convolution on the left. Since the value of $\varphi(a)$ on the trivial coset $1_K \in \mathbb{Z}[K \backslash G]$ allows to recover $a \in \mathbb{Z}[K \backslash G / K]$, the map φ is injective. Moreover, by G -equivariance, any element

$$\alpha \in \mathrm{hom}_{\mathbb{Z}}(\mathbb{Z}[K \backslash G], \mathbb{Z}[K \backslash G])^G$$

is determined by its value $\alpha(1_K)$ on the trivial coset. Now, $\alpha(1_K) = \sum_{Ks \in K \backslash G} a_{Ks} \cdot 1_{Ks}$. But a_{Ks} is constant on double cosets $K \backslash G / K$, since α is G -equivariant and 1_K is fixed by K . This shows that

$$\alpha(1_K) = \sum_{KsK \in K \backslash G / K} a_{KsK} \cdot 1_{KsK}$$

and proves the claim. \square

Note that the augmentation homomorphism $\varepsilon: \mathbb{Z}[G, K] \rightarrow \mathbb{Z}$ which is given by $\varepsilon(1_{KgK}) := \nu(KgK)$ defines a $\mathbb{Z}[G, K]$ -module structure on \mathbb{Z} .

2.3. Homological algebra for the Hecke algebra. Let $\mathcal{K}(G)$ denote the set of compact open subgroups of G partially ordered by inclusion.

A (possibly non-unital) ring R is called *idempotented* if for every finite set of elements $S \subset R$ there is an idempotent $q \in R$ such that $qx = xq = x$ for all $x \in S$. A (left) R -module M is *non-degenerate* if $M = R \cdot M$. It is easy to see that the category of non-degenerate R -modules is an abelian category. For an idempotent $q \in R$ the R -module Rq is projective in this category. Since every element in a non-degenerate R -module M is in the image of a homomorphism $Rq \rightarrow M$ for some idempotent q , the category of non-degenerate R -modules has enough projectives. Similarly there are enough injectives. So the derived functors of the hom and tensor product functors are available, and the usual notions and tools from homological algebra still work for idempotented rings. See [7, Chapter XII]. The same discussion applies for right $\mathcal{H}(G)$ -modules. The Hecke algebra $\mathcal{H}(G)$ is idempotented; it is the union of $1_K \mathcal{H}(G) 1_K = \mathcal{H}(G, K)$, $K \in \mathcal{K}(G)$.

For a discrete subgroup $\Gamma < G$, we may now consider $\mathcal{H}(G)$ as a $\mathcal{H}(G)$ - $\mathbb{C}[\Gamma]$ -bimodule and use it to formulate a suitable Shapiro lemma for homology. Moreover, the induction by $\mathcal{H}(G)$ can be explicitly computed in special cases, for example $\mathcal{H}(G) \otimes_{\mathbb{C}[\Gamma]} \mathbb{C} = \mathcal{H}(G/\Gamma)$, where $\mathcal{H}(G/\Gamma)$ denotes the vector space of locally constant and compactly supported functions on the homogenous space G/Γ .

LEMMA 2.3 (Shapiro). *Let Γ be a discrete subgroup in G and let M be a left Γ -module. There is a natural isomorphism*

$$H_*(\Gamma, M) \xrightarrow{\sim} \mathrm{Tor}_*^{\mathcal{H}(G)}(\mathbb{C}, \mathcal{H}(G) \otimes_{\mathbb{C}[\Gamma]} M).$$

PROOF. It suffices to prove that $\mathcal{H}(G)$ is a flat right $\mathbb{C}[\Gamma]$ -module. For a compact open subgroup $K < G$ the right $\mathbb{C}[\Gamma]$ -module $1_K \mathcal{H}(G)$ is isomorphic to $\mathbb{C}[K \backslash G]$ where $K \backslash G$ is a right Γ -set with finite stabilisers. Such a $\mathbb{C}[\Gamma]$ -module is projective. Since $\mathcal{H}(G)$ is the directed colimit of right $\mathbb{C}[\Gamma]$ -modules $1_K \mathcal{H}(G)$ over $K \in \mathcal{K}(G)$, it is flat. \square

3. Spectral approximation in Hecke algebras

Throughout, let $K < G$ be a compact open subgroup. We normalise the Haar measure ν on G so that $\nu(K) = 1$. For a lattice $\Gamma < G$ we denote by $\nu_{G/\Gamma}$ the finite measure on $X := G/\Gamma$ induced by ν . We do not normalise $\nu_{G/\Gamma}$; its total mass is $\mathrm{covol}_\nu(\Gamma)$, the ν -covolume of Γ .

3.1. Positive functionals on the Hecke algebra.

We denote by

$$\bar{X} := K \backslash X = K \backslash G / \Gamma$$

the countable set of double cosets and note that \bar{X} carries a finite measure which gives the double coset $Ks\Gamma \in K \backslash G / \Gamma$ the measure of its equivalence class $Ks\Gamma$, seen as a subset of X . Note that the canonical map $K \rightarrow Ks\Gamma$, $k \mapsto ks\Gamma$ is a finite covering with fibre of size equal to $|K \cap s\Gamma s^{-1}|$. Hence, we obtain $\nu_{G/\Gamma}(Ks\Gamma) = |K \cap s\Gamma s^{-1}|^{-1}$. Thus, since Γ is a lattice, we conclude

$$(3.1) \quad \sum_{Ks\Gamma} \frac{1}{|K \cap s\Gamma s^{-1}|} = \sum_{Ks\Gamma \in \bar{X}} \nu_{G/\Gamma}(Ks\Gamma) = \nu_{G/\Gamma}(X) = \text{covol}_\nu(\Gamma) < \infty.$$

Further, let \bar{X}_e be the set of double cosets $Ks\Gamma \in \bar{X}$ such that the map $K \ni k \mapsto ks\Gamma \in X$ is injective or, equivalently,

$$K \cap s\Gamma s^{-1} = \{e\}.$$

In particular, for each point $Ks\Gamma \in \bar{X}_e$, the corresponding equivalence class $Ks\Gamma \subseteq X$ has measure $\nu_{G/\Gamma}(Ks\Gamma) = 1$. It follows that the set \bar{X}_e is finite, with cardinality

$$(3.2) \quad |\bar{X}_e| \leq \text{covol}_\nu(\Gamma).$$

More precisely, we obtain

$$|\bar{X}_e| = \nu_{G/\Gamma}(\{s\Gamma \mid K \cap s\Gamma s^{-1} = \{e\}\}).$$

Thus, we see that $\ell^2(K \backslash G / \Gamma, \nu)$ contains an isometric copy $\ell^2(\bar{X}_e)$.

DEFINITION 3.1. *We denote the projection of $\ell^2(K \backslash G / \Gamma, \nu)$ onto the subspace $\ell^2(\bar{X}_e)$ by*

$$P_\Gamma : \ell^2(K \backslash G / \Gamma, \nu) \rightarrow \ell^2(\bar{X}_e).$$

We want to understand in what sense the quotients of the form $K \backslash G / \Gamma$ approximate $K \backslash G$ as Γ varies. Our emphasis is on the non-uniform case where special care is needed. Indeed, we would like to consider a functional

$$\text{tr}_\Gamma : \mathcal{H}(G, K) \rightarrow \mathbb{C},$$

of the form

$$\text{tr}_\Gamma(f) = \frac{1}{\text{covol}_\nu(\Gamma)} \int_{G/\Gamma} \sum_{\gamma \in \Gamma} f(s\gamma s^{-1}) d\nu_{G/\Gamma}(s),$$

which means more concretely

$$\text{tr}_\Gamma(1_{KgK}) := \frac{1}{\text{covol}_\nu(\Gamma)} \int_{G/\Gamma} |KgK \cap s\Gamma s^{-1}| d\nu_{G/\Gamma}(s).$$

If Γ is a uniform lattice, then tr_Γ defines a (usually unnormalised) positive trace on $\mathcal{H}(G, K)$, that can be used to relate the spectral properties of the action of

the Hecke algebra on $\ell^2(K \backslash G / \Gamma, \nu)$ to those of the action on $\ell^2(K \backslash G)$. However, since $\nu_{G/\Gamma}(Ks\Gamma) = |KgK \cap s\Gamma s^{-1}|^{-1}$, the integral for $\text{tr}_\Gamma(1_K)$ will be infinite if the lattice is non-uniform. Thus, this approach is of no use in the general case.

In order to overcome this problem, we have to perform a renormalization procedure that concentrates on double cosets of full measure and control its defect on small double cosets. Note that the subspace of $L^2(G/\Gamma, \nu_{G/\Gamma})$ formed of K -invariant functions, which is spanned by a set of orthogonal functions $\{1_{Ks\Gamma} \mid Ks\Gamma \in \bar{X}\}$ is just $\ell^2(K \backslash G / \Gamma, \nu)$, i.e. the weighted ℓ^2 -space on the set $K \backslash G / \Gamma$ with weights as given above. The Hilbert space $\ell^2(K \backslash G / \Gamma, \nu)$ is endowed with an action

$$\pi_\Gamma: \mathcal{H}(G, K) \rightarrow B(\ell^2(K \backslash G / \Gamma, \nu))$$

given by convolution. We have

$$\langle \pi_\Gamma(1_{KgK})(1_{Kh\Gamma}), 1_{Ks\Gamma} \rangle = \langle 1_{KgK} * 1_{Kh\Gamma}, 1_{Ks\Gamma} \rangle = \nu(KgK \cap s\Gamma h^{-1}K) \in \mathbb{Z},$$

where integrality follows from $KgK \cap s\Gamma h^{-1}K$ being a finite union of right K -cosets. We record for later:

REMARK 3.2. *Hence the action of 1_{KgK} on $\ell^2(K \backslash G / \Gamma, \nu)$ is given, with respect to the canonical basis consisting of indicator functions, by an infinite matrix with integer entries and only finitely many non-zero entries in each row and column.*

DEFINITION 3.3. *The map*

$$\varphi_\Gamma: \mathcal{H}(G, K) \rightarrow \mathbb{C}, \quad \varphi_\Gamma(T) := \frac{\sum_{Ks\Gamma \in \bar{X}} \langle \pi_\Gamma(T)(1_{Ks\Gamma}), 1_{Ks\Gamma} \rangle}{\text{covol}_\nu(\Gamma)}.$$

defines a unital and positive linear functional on $\mathcal{H}(G, K)$. The map

$$\varphi_\Gamma^e(1_{KgK}) = \frac{\sum_{Ks\Gamma \in \bar{X}_e} \langle \pi_\Gamma(T)(1_{Ks\Gamma}), 1_{Ks\Gamma} \rangle}{\text{covol}_\nu(\Gamma)} = \frac{\sum_{Ks\Gamma \in \bar{X}_e} \langle P_\Gamma \pi_\Gamma(T) P_\Gamma^*(1_{Ks\Gamma}), 1_{Ks\Gamma} \rangle}{\text{covol}_\nu(\Gamma)}.$$

defines a (possibly non-unital) positive linear functional on $\mathcal{H}(G, K)$.

One easily verifies that

$$(3.3) \quad \varphi_\Gamma(1_{KgK}) = \frac{\sum_{Ks\Gamma \in \bar{X}} \nu(KgK \cap s\Gamma s^{-1}K) \nu_{G/\Gamma}(Ks\Gamma)}{\text{covol}_\nu(\Gamma)} \in [0, 1].$$

A similar identity holds for $\varphi_\Gamma^e(1_{KgK})$ with the sum only running over $Ks\Gamma \in \bar{X}_e$. This time, we obtain a positive functional φ_Γ resembling the spectral properties of the action on $\ell^2(K \backslash G / \Gamma, \nu)$, no matter if Γ is uniform or not.

The functionals tr and ϕ_{Γ_i} and ϕ_Γ^e (by first taking the matrix trace and then the functional) as well as other notions discussed so far extend to matrix algebras $M_n(\mathcal{H}(G, K))$ for $n \in \mathbb{N}$ – and we will use the notation introduced so far unchanged in the setting of matrix algebras.

3.2. Spectral approximation for Farber sequences. We are now finished with our preparations and will proceed by stating and proving the first main theorem. Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a sequence of lattices in G . Let $T \in M_n(\mathcal{H}(G, K))$ be self-adjoint. By the Riesz representation theorem there is a unique Borel measure μ_T , called the *spectral measure* with respect to tr , on \mathbb{R} such that

$$\text{tr}(T^k) = \int_{\mathbb{R}} t^k d\mu_T(t).$$

Similarly, one defines the spectral measures $\mu_{T,i}$ and $\mu_{T,i}^e$ with respect to ϕ_{Γ_i} and $\phi_{\Gamma_i}^e$. The measure μ_T is supported in the interval $[-\|\pi(T)\|, \|\pi(T)\|]$. The measures $\mu_{T,i}$ and $\mu_{T,i}^e$ are supported in the interval $[-\|\pi_{\Gamma_i}(T)\|, \|\pi_{\Gamma_i}(T)\|]$. Since the operator norms $\pi(T)$ and $\pi_{\Gamma_i}(T)$ are bounded by the maximum of the (obvious) ℓ^1 -norms of the entries of T times n^2 (cf. [20, Lemma 13.33 on p. 470]), all three measures are supported on the compact interval $[-c, c]$ with

$$(3.4) \quad c := \max\{\|T_{i,j}\|_1 \mid i, j \in \{1, \dots, n\}\} \cdot n^2.$$

Moreover, μ_T and $\mu_{T,i}$ are probability measures while the total mass of $\mu_{T,i}^e$ may be less than 1.

THEOREM 3.4. *Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a Farber sequence of lattices in G . Let $T \in M_n(\mathbb{Z}[G, K])$ be a self-adjoint element.*

- (i) *The sequences of measures $(\mu_{T,i})$ and $(\mu_{T,i}^e)$ both weakly converge to μ_T .*
- (ii) *We have $\mu_T(\{0\}) = \lim_{i \rightarrow \infty} \mu_{T,i}(\{0\}) = \lim_{i \rightarrow \infty} \mu_{T,i}^e(\{0\})$.*
- (iii) *Assume in addition that $T = S^*S$ is positive. Then we have*

$$\mu_T(\{0\}) = \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{C}} \ker(P_{\Gamma_i} \pi_{\Gamma_i}(T) P_{\Gamma_i}^*)}{\text{covol}_{\nu}(\Gamma_i)} \leq \liminf_{i \rightarrow \infty} \frac{\dim_{\mathbb{C}} \ker(\pi_{\Gamma_i}(T))}{\text{covol}_{\nu}(\Gamma_i)}$$

with equality provided that the sequence $(\Gamma_i)_{i \in \mathbb{N}}$ is uniformly discrete.

PROOF. (i) Every ϕ_{Γ_i} is unital, so $\phi_{\Gamma_i}(1_K) = 1$. For $g \in G$ we have

$$\nu(KgK \cap s\Gamma_i s^{-1}K) = \nu((KgK \cap s\Gamma_i s^{-1}) \cdot K) \leq \begin{cases} \nu(KgK) & \text{if } KgK \cap s\Gamma_i s^{-1} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, using Equation 3.3,

$$\begin{aligned} \phi_{\Gamma_i}(1_{KgK}) &\leq \nu(KgK) \cdot \sum_{\substack{Ks\Gamma_i \in K \backslash G / \Gamma_i \\ KgK \cap s\Gamma_i s^{-1} \neq \emptyset}} \frac{\nu_{G/\Gamma_i}(Ks\Gamma_i)}{\text{covol}_{\nu}(\Gamma_i)} \\ &= \nu(KgK) \cdot \frac{\nu_{G/\Gamma_i}(\{s\Gamma_i \mid KgK \cap s\Gamma_i s^{-1} \neq \emptyset\})}{\text{covol}_{\nu}(\Gamma_i)} \\ &= \nu(KgK) \cdot \mu_{\Gamma_i}(\{H \in \text{Sub}_G \mid H \cap KgK \neq \emptyset\}) \end{aligned}$$

For $g \notin K$ the latter tends to zero as $i \rightarrow \infty$ by the Farber condition as KgK can be covered by finitely many non-trivial K -cosets. Since ϕ_{Γ_i} is unital we obtain that

$$\lim_{i \rightarrow \infty} \phi_{\Gamma_i}(1_{KgK}) = \text{tr}(1_{KgK})$$

for every double coset KgK . Since $0 \leq \phi_{\Gamma_i}^e(1_{KgK}) \leq \phi_{\Gamma_i}(1_{KgK})$ we obtain also that

$$\lim_{i \rightarrow \infty} \phi_{\Gamma_i}^e(1_{KgK}) = \text{tr}(1_{KgK}) = 0 \quad \text{for } g \notin K.$$

And we have

$$\begin{aligned} \phi_{\Gamma_i}^e(1_K) &= \sum_{\substack{Ks\Gamma_i \\ K \cap s\Gamma_i s^{-1} = \{e\}}} \frac{\nu(Ks\Gamma_i)}{\text{covol}_\nu(\Gamma_i)} = \frac{|\{Ks\Gamma_i \in K \backslash G/\Gamma_i \mid K \cap s\Gamma_i s^{-1} = \{e\}\}|}{\text{covol}_\nu(\Gamma_i)} \\ &= \mu_{\Gamma_i}(\{H \in \text{Sub}_G \mid H \cap s\Gamma_i s^{-1} = \{e\}\}). \end{aligned}$$

By the Farber condition $\phi_{\Gamma_i}^e(1_K)$ tends to $\text{tr}(1_K) = 1$ as $i \rightarrow \infty$. This implies that the spectral measures $\mu_{T,i}$ and $\mu_{T,i}^e$ both converge to μ_T in moments. Since all the measures are supported on the same compact interval, $\mu_{T,i}$ and $\mu_{T,i}^e$ weakly converge to μ_T .

(ii) By the Portmanteau theorem weak convergence is equivalent to

$$\mu_T(E) \geq \limsup_i \mu_{T,i}(F), \quad \mu_T(U) \leq \liminf_i \mu_{T,i}(U).$$

for any closed subset $F \subset \mathbb{R}$ and any open subset $U \subset \mathbb{R}$. Similarly for $\mu_{T,i}^e$. In particular, we have $\limsup \mu_{T,i}(\{0\}) \leq \mu_T(\{0\})$ and $\limsup \mu_{T,i}^e(\{0\}) \leq \mu_T(\{0\})$. Clearly, $\liminf \mu_{T,i}^e(\{0\}) \leq \liminf \mu_{T,i}(\{0\})$. So it remains to show that

$$(3.5) \quad \liminf_{i \rightarrow \infty} \mu_{T,i}^e(\{0\}) \geq \mu_T(\{0\}).$$

The basic mechanism for proving this builds on the integrality of T and goes back to the work of Lück [18]. We apply this mechanism in our setting.

Let $c > 0$ be as in (3.4). Fix $i \in \mathbb{N}$. It is clear that $P_{\Gamma_i} \pi_{\Gamma_i}(T) P_{\Gamma_i}^*$ is a finite-dimensional self-adjoint operator. Let e_1, \dots, e_l be its eigenvalues written with multiplicity and ordered by increasing absolute value, and let e_{m+1} be the first non-zero eigenvalue. Since $P_{\Gamma_i} \pi_{\Gamma_i}(T) P_{\Gamma_i}^*$ acts as an $l \times l$ matrix with integer entries on $\ell^2(\bar{X}_e)$, we have $|e_{m+1} \cdots e_l| \geq 1$ and $|e_l| \leq c$. By (3.2) we have $l \leq |\bar{X}_e| \leq \text{covol}_\nu(\Gamma_i)$. Fix an $\varepsilon > 0$ and let $\delta := |\{i \mid e_i \in (-\varepsilon, \varepsilon) \setminus \{0\}\}|$. Then $1 \leq \varepsilon^\delta \cdot c^l$. Note that $\mu_{T,i}^e((-\varepsilon, \varepsilon) \setminus \{0\})$ is the matrix trace, normalised by $\text{covol}_\nu(\Gamma_i)$, of the projection onto the sum of eigenspaces of $P_{\Gamma_i} \pi_{\Gamma_i}(T) P_{\Gamma_i}^*$ corresponding to eigenvalues in $(-\varepsilon, \varepsilon) \setminus \{0\}$. Hence $\mu_{T,i}^e((-\varepsilon, \varepsilon) \setminus \{0\}) = \delta / \text{covol}_\nu(\Gamma_i)$. It follows that

$$\mu_{T,i}^e((-\varepsilon, \varepsilon) \setminus \{0\}) = \frac{\delta}{\text{covol}_\nu(\Gamma_i)} \leq \frac{\delta}{l} \leq \frac{\log c}{|\log \varepsilon|}.$$

Since $i \in \mathbb{N}$ was arbitrary, we conclude that, for any $\varepsilon > 0$,

$$\begin{aligned} \liminf_{i \rightarrow \infty} \mu_{T,i}^e(\{0\}) &\geq \liminf_{i \rightarrow \infty} \mu_{T,i}^e((-\varepsilon, \varepsilon)) - \frac{\log c}{|\log \varepsilon|} \\ &\geq \mu_T((-\varepsilon, \varepsilon)) - \frac{\log c}{|\log \varepsilon|} \\ &\geq \mu_T(\{0\}) - \frac{\log c}{|\log \varepsilon|}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we conclude (3.5).

(iii) By definition, $\mu_{T,i}^e(\{0\})$ is the matrix trace, normalised by $\text{covol}_\nu(\Gamma_i)$, of the projection of $\ell^2(\bar{X}_e)$ onto the kernel of $P_{\Gamma_i} \pi_{\Gamma_i}(T) P_{\Gamma_i}^*$. Hence $\mu_{T,i}^e(\{0\})$ is just the vector space dimension of $\ker(P_{\Gamma_i} \pi_{\Gamma_i}(T) P_{\Gamma_i}^*)$ normalised by $\text{covol}_\nu(\Gamma_i)$. By positivity of $\pi_{\Gamma_i}(T)$ we have

$$\dim_{\mathbb{C}} \ker(P_{\Gamma_i} \pi_{\Gamma_i}(T) P_{\Gamma_i}^*) \leq \dim_{\mathbb{C}} \ker(\pi_{\Gamma_i}(T)),$$

thus the stated inequality follows. If $(\Gamma_i)_{i \in \mathbb{N}}$ is uniformly discrete, then $P_{\Gamma_i} = \text{id}$ for i sufficiently large, and we get equality. This finishes the proof. \square

4. L^2 -Betti numbers of totally disconnected groups

In this section we review the definition of L^2 -Betti numbers of totally disconnected groups from [22] and provide some additional tools. The algebraic approach pioneered by Lück [20] has found various applications in ergodic theory, algebra and geometry, see [25, 28, 29, 33, 34]. We conclude the proof of Theorem 1.3 in Subsection 4.4.

4.1. The group von Neumann algebra. We denote by

$$\lambda: \mathcal{H}(G) \rightarrow B(L^2(G, \nu))$$

the left-regular representation and by $\rho: \mathcal{H}(G) \rightarrow B(L^2(G, \nu))$ the right-regular representation – defined by left and right convolution, respectively. The *group von Neumann algebra* of G is defined to be the weak closure

$$L(G) = \overline{\lambda(\mathcal{H}(G))}^w.$$

The corresponding closure of the image of ρ is naturally anti-isomorphic to $L(G)$, so that $L^2(G, \nu)$ becomes a $L(G)$ -bimodule in a natural way.

Let $\mathcal{H}(G)_+$ and $L(G)_+$ denote the subsets of positive elements x^*x , respectively. The trace tr in Definition 2.1 is independent of the Haar measure, but the algebra structure of $\mathcal{H}(G)$ depends on the choice of a Haar measure, being defined in terms of convolution. The (restricted) trace $\text{tr}|_{\mathcal{H}(G)_+}: \mathcal{H}(G)_+ \rightarrow [0, \infty)$ extends to a faithful normal semifinite trace $\text{tr}: L(G)_+ \rightarrow [0, \infty]$. Further, it extends to a faithful normal semifinite trace on the von Neumann algebra of $n \times n$ -matrices over $L(G)$

which we denote by the same symbol. In particular, $L(G)$ and $M_n(L(G))$ are semifinite von Neumann algebras. Since the linear span of positive elements with finite trace is dense in $L(G)$, tr induces a densely defined, faithful, positive tracial weight on $L(G)$.

Furthermore, if $p \in L(G)$ is a projection with $\text{tr}(p) < \infty$, then $pL(G)p$ is a finite von Neumann algebra with the restriction of tr as finite trace. We usually normalise this trace with $1/\text{tr}(p)$.

We refer to [27] for a more detailed survey of the above notions.

4.2. The dimension for modules over the group von Neumann algebra. We review the definition of dimension of arbitrary right $L(G)$ -modules in [22] which generalizes the corresponding work of Lück for discrete groups.

The dimension of a finitely generated projective (right) $L(G)$ -module $P = pL(G)^n$ where p is a projection in $M_n(L(G))$ is defined as

$$\dim_{(L(G), \text{tr})}(P) := \text{tr}(p) \in [0, \infty].$$

For an arbitrary $L(G)$ -module M one defines

$$\dim_{(L(G), \text{tr})}(M) := \sup\{\dim_{(L(G), \text{tr})}(P) \mid P \subset M \text{ f.g. proj. submodule}\} \in [0, \infty].$$

This dimension is additive for short exact sequences and continuous with respect to ascending unions of modules [22, Theorems B.22 and B.23].

REMARK 4.1. *If q is a projection in $M_n(L(G))$ and $M = qL^2(G, \nu)^n$ is the image of q then $\dim_{(L(G), \text{tr})}(M) = \text{tr}(q)$ by [22, Theorem B.25]. If $K \subset G$ is a compact open subgroup and p_K denotes the projection onto the left K -invariant vectors in $L^2(G, \nu)$, then we obtain*

$$\dim_{(L(G), \text{tr})}(\ell^2(K \backslash G)) = \dim_{(L(G), \text{tr})}(p_K L^2(G, \nu)) = \text{tr}(p_K) = \frac{1}{\nu(K)}.$$

Next we introduce some tools that are needed to express L^2 -Betti numbers of totally disconnected groups via the dimension over a finite von Neumann algebra (see Lemma 5.1).

REMARK 4.2. *If $K < K'$ is an inclusion of compact open subgroup in G , then the projections satisfy $p_{K'} \leq p_K$. For two projections p, q in a von Neumann algebra we write $p \sim q$ if p, q are Murray-von Neumann equivalent, and we write $p \preceq q$ if there is $\tilde{p} \sim p$ such that $\tilde{p} \leq q$. If the subgroup K is subconjugated to K' , then $p_{K'} \preceq p_K$.*

DEFINITION 4.3. *Let M be a $L(G)$ -module. Let $p \in L(G)$ be a projection. The support $s(x) \in L$ of an element $x \in M$ is the smallest projection $s(x) \in L$ such that $xs(x) = x$. We say that M is p -truncated if $s(x) \preceq p$ holds for every $x \in M$.*

REMARK 4.4. *The class of p -truncated modules over $L(G)$ is closed under taking submodules and homomorphic images. The prototypical case of a p -truncated $L(G)$ -module is $pL(G)$: Let $x \in pL(G)$. Let $s_l(x)$ be the smallest projection in $L(G)$ with $s_l(x)x = x$. Clearly, we have $s_l(x) \leq p$. By polar decomposition, $s(x) \sim s_l(x)$ (cf. [31, Proposition 1.5 on p. 292]).*

The following lemma is crucial for relating various definitions of L^2 -Betti numbers that exist in the literature.

LEMMA 4.5. *Let $p \in L(G)$ be a projection with $\text{tr}(p) < \infty$. Let M be a p -truncated $L(G)$ -module. We have*

$$\dim_{(L(G), \text{tr})}(M) = \text{tr}(p) \cdot \dim_{(pL(G)p, \text{tr}(p)^{-1} \text{tr})}(pM).$$

PROOF. First we show the assertion for finitely generated projective modules of finite $L(G)$ -dimension. Let $q \in M_n(L(G))$ be a projection such that the $L(G)$ -module $qL(G)^n$ is p -truncated. Let $e_i \in L(G)^n$ be the i -th standard basis vector. Let $x_i = qe_i \in P$. By assumption $s(x_i) \preceq p$. Since $s(x_i)$ and p are finite projections there are unitaries $u_i \in L$ such that $s(x_i) \leq u_i^* p u_i$ [31, Proposition 1.38 on p. 304]. Hence

$$q \cdot \text{diag}(u_1^* p u_1, \dots, u_n^* p u_n) = q.$$

This implies

$$q \preceq \text{diag}(p, \dots, p) \sim \text{diag}(u_1^* p u_1, \dots, u_n^* p u_n).$$

Let $q' \sim q$ a projection in $M_n(L(G))$ such that $q' \leq \text{diag}(p, \dots, p)$. Then

$$\begin{aligned} \dim_{(pL(G)p, \text{tr}(p)^{-1} \text{tr})}(qL(G)^n p) &= \dim_{(pL(G)p, \text{tr}(p)^{-1} \text{tr})}(q' L(G)^n p) \\ &= \dim_{(pL(G)p, \text{tr}(p)^{-1} \text{tr})}(q' p L(G)^n p) \\ &= \text{tr}(q') \text{tr}(p)^{-1} \\ &= \text{tr}(q) \text{tr}(p)^{-1} \\ &= \dim_{(L(G), \text{tr})}(qL(G)^n) \text{tr}(p)^{-1}. \end{aligned}$$

Let M be now an arbitrary $L(G)$ -module. For every finitely generated projective submodule P of M , Pp is a submodule of Mp whose dimension with respect to $pL(G)p$ is $\dim_{(L(G), \text{tr})}(P) \text{tr}(p)^{-1}$. This implies the \leq -inequality in the statement of the lemma. For the \geq -inequality we refer to the proof of [22, B.35 Theorem]. \square

4.3. L^2 -Betti numbers. The L^2 -Betti numbers of a locally compact, separable, and unimodular group are defined as

$$\beta_n^{(2)}(G, \nu) := \dim_{(L(G), \text{tr})} H_c^n(G, L^2(G)),$$

where H_c^n denotes the continuous group cohomology [22]. In the case where G is totally disconnected, it is shown in [22] that

$$\beta_n^{(2)}(G, \nu) := \dim_{(L(G), \text{tr})} \text{Tor}_*^{\mathcal{H}(G)}(\mathbb{C}, L_\infty^2(G, \nu)).$$

Here

$$L_\infty^2(G) := \bigcup_{K \in \mathcal{K}(G)} \ell^2(K \backslash G, \nu) \subset L^2(G, \nu)$$

is the vector space of smooth vectors which is naturally a $\mathcal{H}(G)$ - $L(G)$ -bimodule.

From a topological model X of G we obtain projective resolutions of \mathbb{C} in the category of (non-degenerate) $\mathcal{H}(G)$ -modules which compute the above Tor group: Upon taking inverses we may assume that G acts from the right on X . The n -skeleton $X^{(n)}$ is built from $X^{(n-1)}$ by attaching G -orbits of n -cells according to pushouts of G -spaces of the form:

$$(4.1) \quad \begin{array}{ccc} \bigsqcup_{U \in \mathcal{F}_n} U \backslash G \times S^{n-1} & \longrightarrow & X^{(n-1)} \\ \downarrow & & \downarrow \\ \bigsqcup_{U \in \mathcal{F}_n} U \backslash G \times D^n & \longrightarrow & X^{(n)} \end{array}$$

Here \mathcal{F}_n is the set of n -cells, which we loosely identify with a multi-set of representatives of conjugacy classes of stabilisers of n -cells. Each coset space $U \backslash G$ is discrete. Let us fix a choice of pushouts, which is not part of the data of a cellular complex and corresponds to an equivariant choice of orientations for the cells. The horizontal maps induce an isomorphism in relative homology by excision. Let $C_*(X)$ be the cellular chain complex. Every chain group $C_n(X)$ is a *discrete G -module*, i.e. an abelian group with a right G -action by homomorphisms such that the stabiliser of any element is open. The differentials are G -equivariant. We obtain isomorphisms of discrete G -modules:

$$(4.2) \quad \bigoplus_{U \in \mathcal{F}_n} \mathbb{Z}[G/U] \cong H_n\left(\bigsqcup_{U \in \mathcal{F}_n} G/U \times (D^n, S^{n-1})\right) \xrightarrow{\cong} H_n(X^{(n)}, X^{(n-1)}) \stackrel{\text{def}}{=} C_n(X)$$

REMARK 4.6. For any discrete G -module M , $\mathbb{C} \otimes_{\mathbb{Z}} M$ is naturally a right $\mathcal{H}(G)$ -module via

$$mf = \int_G f(g)mgd\nu(g) \quad \text{for } f \in \mathcal{H}(G) \text{ and } m \in \mathbb{C} \otimes_{\mathbb{Z}} M.$$

This has to be understood as follows: If $U < G$ is the stabiliser of m , then read the integral as the finite sum $\sum_{g \in G/U} (\int_U f(gu)d\nu(u))mg$. Any homomorphism of discrete G -modules becomes thus a homomorphism of $\mathcal{H}(G)$ -modules.

Hence the cellular chain complex $C_*(X; \mathbb{C})$ with complex coefficients is a chain complex of $\mathcal{H}(G)$ -modules. Since $\mathbb{C}[U \backslash G]$ is isomorphic to $p_U \mathcal{H}(G)$ when regarded

as a $\mathcal{H}(G)$ -module, $C_*(X; \mathbb{C})$ is a projective resolution of $\mathbb{C} \cong H_0(X)$ in the category of non-degenerate right $\mathcal{H}(G)$ -modules. Hence we can conclude:

REMARK 4.7. *Let X be a topological model of G . Then*

$$\beta_n^{(2)}(G, \nu) = \dim_{(L(G), \text{tr})} H_n(C_*(X; \mathbb{C}) \otimes_{\mathcal{H}(G)} L_\infty^2(G, \nu)).$$

DEFINITION 4.8. *If G has a cocompact topological model, then the Euler characteristic of G with respect to the Haar measure ν is – in reference to the setting in (4.2) – defined as*

$$\chi(G, \nu) = \sum_{p \geq 0} (-1)^p \sum_{U \in \mathcal{F}_n} \mu(U)^{-1}.$$

By the same argument as in [20, Theorem 1.35], which is solely based on the additivity of the dimension, we obtain the Euler-Poincaré formula:

THEOREM 4.9. *If G has a cocompact topological model, then*

$$\chi(G, \nu) = \sum_{p \geq 0} (-1)^p \beta_p^{(2)}(G, \nu).$$

Again, $\chi(G, \nu)$ denotes the appropriate notion of Euler characteristic in this context, where a G -cell of the form $U \backslash G \times D^n$ is counted $(-1)^n \cdot \nu(U)^{-1}$.

4.4. Conclusion of proof of Theorem 1.3. We retain the notation of Theorem 1.3. Let X be a topological model of G whose $(n+1)$ -skeleton is cocompact, i.e. has only finitely many G -equivariant cells up to dimension $(n+1)$. We pick G -pushouts as in (4.1). Let K be the (finite) intersection of the corresponding stabiliser subgroups $U \in \mathcal{F}_i, i \in \{0, \dots, n+1\}$. Without loss of generality we normalise the Haar measure so that $\nu(K) = 1$.

For a compact open subgroup $L > K$ the discrete G -module $\mathbb{Z}[K \backslash G]$ is isomorphic to $[L : K]$ many copies of $\mathbb{Z}[L \backslash G]$. Each $C_k(X)$ with $k \in \{0, \dots, n+1\}$ is a finite sum of discrete G -modules of the form $\mathbb{Z}[L \backslash G]$ for $K < L$. By a standard method from homological algebra we may add short exact chain complexes of the form $\mathbb{Z}[L \backslash G] \xrightarrow{\text{id}} \mathbb{Z}[L \backslash G], K < L$, to the resolution $C_*(X)$ starting in degree 0 up to degree $(n+1)$ so that we obtain a resolution of \mathbb{Z} by discrete G -modules F_* with differentials δ_* where each $F_k, k \in \{0, \dots, n+1\}$, is a finite sum of copies of $\mathbb{Z}[K \backslash G]$. Let $(F_*^\mathbb{C}, \delta_*^\mathbb{C})$ be the complexification of (F_*, δ_*) . It follows from the discussion in the previous subsection that $F_*^\mathbb{C}$ yields a projective $\mathcal{H}(G)$ -resolution. Accordingly the L^2 -Betti numbers of G are computed as the $L(G)$ -dimensions of the homology of $F_*^\mathbb{C} \otimes_{\mathcal{H}(G)} L_\infty^2(G, \nu)$. For each degree $k \in \{0, \dots, n+1\}$ let a_k be the number of summands $\mathbb{Z}[K \backslash G]$ in F_k . Then we have the canonical isomorphisms

of $L(G)$ -modules:

$$\begin{aligned} F_k^{\mathbb{C}} \otimes_{\mathcal{H}(G)} L_{\infty}^2(G, \nu) &\cong \mathbb{C}[K \backslash G]^{a_k} \otimes_{\mathcal{H}(G)} L_{\infty}^2(G, \nu) \cong (p_K \mathcal{H}(G))^{a_k} \otimes_{\mathcal{H}(G)} L_{\infty}^2(G, \nu) \\ &\cong (p_K L_{\infty}^2(G, \nu))^{a_k} \\ &\cong L^2(K \backslash G)^{a_k}. \end{aligned}$$

By Lemma 2.2 the differentials δ_* in degrees $\leq n+1$ are given by convolution with finite-dimensional matrices over $\mathbb{Z}[G, K]$. Hence the same is true for the differentials $\delta_*^{(2)} = \delta_*^{\mathbb{C}} \otimes \text{id}$ of $F_*^{\mathbb{C}} \otimes_{\mathcal{H}(G)} L_{\infty}^2(G, \nu)$. The Hilbert adjoint of $\delta_k^{(2)}$ is induced from the formal adjoint of δ_k which also comes from a matrix with entries in $\mathbb{Z}[G, K]$. Hence the combinatorial Laplacian $\Delta_k = \delta_{k+1}^{(2)} (\delta_{k+1}^{(2)})^* + (\delta_k^{(2)})^* \delta_k^{(2)}$ in degrees $k \in \{0, \dots, n\}$ is given by an element $T_k \in M_{a_k}(\mathbb{Z}[G, K])$, or in the notation of Subsection 2.1, $\Delta_k = \pi(T_k)$. We then have the analogue of the usual L^2 -Hodge theorem, which yields

$$\beta_k^{(2)}(G, \nu) = \dim_{(L(G), \text{tr})} \ker(\Delta_k : L^2(K \backslash G)^{a_k} \rightarrow L^2(K \backslash G)^{a_k}).$$

In the sequel we refer a lot to the notation of Subsection 3.1. Let now $\Gamma_i < G$ be a lattice from the Farber sequence. By Lemma 2.3,

$$H_k(\Gamma_i, \mathbb{C}) \cong \text{Tor}_k^{\mathcal{H}(G)}(\mathbb{C}, \mathcal{H}(G) \otimes_{\mathbb{C}[\Gamma_i]} \mathbb{C}) = H_k(F_*^{\mathbb{C}} \otimes_{\mathbb{C}[\Gamma_i]} \mathbb{C}).$$

Note that $F_k^{\mathbb{C}} \otimes_{\mathbb{C}[\Gamma_i]} \mathbb{C}$ is canonically isomorphic to $\mathbb{C}[K \backslash G / \Gamma_i]^{a_k}$. We can consider the combinatorial Laplacian on $F_k^{\mathbb{C}} \otimes_{\mathbb{C}[\Gamma_i]} \mathbb{C} = \mathbb{C}[K \backslash G / \Gamma_i]^{a_k}$ and on its completion $\ell^2(K \backslash G / \Gamma_i, \nu)^{a_k}$ (with the weighted measure on $K \backslash G / \Gamma$). We denote by $\Delta_k^{\Gamma_i}$ the combinatorial Laplacian on the completion. The set X_e of elements in $K \backslash G / \Gamma$ with measure 1 is finite and we have the inclusions

$$\mathbb{C}[X_{e,i}]^{a_k} \subset \mathbb{C}[K \backslash G / \Gamma_i]^{a_k} \subset \ell^2(K \backslash G / \Gamma_i, \nu)^{a_k}$$

As in Subsection 3.1 we write $\mathbb{C}[X_{e,i}]$ as $\ell^2(X_e)$ to indicate that as a subspace of the Hilbert space $\ell^2(K \backslash G / \Gamma, \nu)$ the set $X_{e,i}$ is a Hilbert basis of $\mathbb{C}[X_{e,i}]$. Let $T_k \in M_{a_k}(\mathbb{Z}[G, K]) \subset M_{a_k}(\mathcal{H}(G; K))$ be the matrix representing Δ_k . We have

$$\Delta_k^{\Gamma_i} = \pi_{\Gamma_i}(T_k).$$

Moreover,

$$\ker(P_{\Gamma_i} \Delta_k^{\Gamma_i} P_{\Gamma_i}^*) = \ker(\Delta_k^{\Gamma_i}) \cap \ell^2(X_{e,i})^{a_k} = \ker(\Delta_k^{\Gamma_i}|_{\ell^2(X_{e,i})^{a_k}}).$$

We have

$$(4.3) \quad \dim_{\mathbb{C}} \ker(P_{\Gamma_i} \Delta_k^{\Gamma_i} P_{\Gamma_i}^*) \leq \beta_n(\Gamma_i),$$

since any element in $\ker(\Delta_k^{\Gamma_i}) \cap \ell^2(X_{e,i})^{a_k}$ is obviously a cycle in $F_*^{\mathbb{C}} \otimes_{\mathbb{C}[\Gamma_i]} \mathbb{C}$ and at the same time orthogonal to the image of the next differential, thus gives rise to

a non-zero class in homology. According to Theorem 3.4 we have

$$\begin{aligned}
\lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{C}} \ker(P_{\Gamma_i} \Delta_k^{\Gamma_i} P_{\Gamma_i}^*)}{\text{covol}_{\nu}(\Gamma_i)} &= \mu_{T_k}(\{0\}) \\
&= \dim_{(L(G), \text{tr})} \ker \pi(T_k) \\
&= \dim_{(L(G), \text{tr})} \ker \Delta_k \\
&= \beta_n^{(2)}(G, \nu).
\end{aligned}$$

By (4.3) this implies immediately that

$$\beta_n^{(2)}(G, \nu) \leq \liminf_{i \rightarrow \infty} \frac{\beta_n(\Gamma_i)}{\text{covol}_{\nu}(\Gamma_i)}.$$

with equality in case of a uniformly discrete Farber sequence of lattices. This proves Theorem 1.3.

5. Applications and examples of groups

5.1. Automorphism groups of buildings. L^2 -Betti numbers of automorphism groups of buildings associated to BN-pairs were intensively studied and computed by Davis, Dymara, Januszkiewicz and Okun [9, 11]. In this subsection we relate our treatment of L^2 -Betti numbers to theirs.

LEMMA 5.1. *Let X be a topological model of G . Let $B < G$ be a compact open subgroup such that for every $x \in X^{(n+1)}$ there is $g \in G$ with $gBg^{-1} \subset G_x$. We normalise the Haar measure ν so that $\nu(B) = 1$. Then with respect to the normalised trace on the finite von Neumann algebra $p_B L(G) p_B$ we have*

$$\begin{aligned}
\beta_i^{(2)}(G, \nu) &= \dim_{p_B L(G) p_B} H_i(G, \ell^2(G/B)) \\
&= \dim_{p_B L(G) p_B} \text{Tor}_i^{\mathbb{Z}[G, B]}(\mathbb{Z}, \ell^2(B \backslash G/B, \nu))
\end{aligned}$$

for $i \in \{0, \dots, n\}$.

PROOF. The cellular chain complex $C_* = C_*(X; \mathbb{C})$ with complex coefficients is a (right) projective $\mathcal{H}(G)$ -resolution of \mathbb{C} . Each chain group $C_i \otimes_{\mathcal{H}(G)} L_{\infty}^2(G)$ for $i \in \{0, \dots, n+1\}$ is by assumption a direct sum of (right) $L(G)$ -modules of type $p_K L_{\infty}^2(G)$ where $K < G$ is a compact open subgroup such that B is subconjugated to K . Hence by Remarks 4.2 and 4.4 the chain complex and thus $H_*(G, L_{\infty}^2(G))$ are p_B -truncated up to degree n . By Lemma 4.5 we have

$$\beta_i^{(2)}(G, \nu) = \dim_{p_B L(G) p_B} (H_i(G, L_{\infty}^2(G)) p_B)$$

in this range. Since p_B is an idempotent we can pull p_B inside by exactness and obtain the first equality of the lemma. For the second equality note that $C_i p_B$ is a projective $\mathbb{Z}[G; B]$ -module for $i \in \{0, \dots, n+1\}$. Since C_i is a direct sum of

modules of type $\mathbb{Z}[K \backslash G]$ with B being subconjugated to K (see above) it suffices to prove that $\mathbb{Z}[K \backslash G]p_B = \mathbb{Z}[K \backslash G/B]$ is a projective $\mathbb{Z}[G; B]$ -module: Let $g \in B$ be such that $B \subset gKg^{-1}$. Then

$$\begin{aligned} \mathbb{Z}[K \backslash G]p_B &\cong \mathbb{Z}[gKg^{-1} \backslash G]p_B \cong p_{gKg^{-1}}\mathcal{H}(G)p_B \\ &= p_{gKg^{-1}}p_B\mathcal{H}(G)p_B \\ &= p_{gKg^{-1}}\mathbb{Z}[G; B]. \end{aligned}$$

Furthermore, one easily verifies that

$$\mathbb{Z}[K \backslash G]p_B \otimes_{\mathbb{Z}[G; B]} \ell^2(B \backslash G/B, \nu) \cong \mathbb{Z}[K \backslash G] \otimes_{\mathcal{H}(G)} \ell^2(G/B).$$

Thus we have

$$C_i p_B \otimes_{\mathbb{Z}[G; B]} \ell^2(B \backslash G/B, \nu) \cong C_i \otimes_{\mathcal{H}(G)} \ell^2(G/B)$$

for $i \in \{0, \dots, n+1\}$ from which we conclude the second equality in the statement. \square

Next, we will consider a totally disconnected group G coming from a BN -pair (G, B, N, S) .

LEMMA 5.2. *Let (G, B, N, S) be a BN -pair. Let $p_B \in L(G)$ be the projection associated to compact open subgroup $B < G$. Then $H_i(G, L^2(G))$ is p_B -truncated for every $i \geq 0$.*

The geometric realisation X of the building of G (in the sense of [10]) is a $\text{CAT}(0)$ -space on which G acts with compact-open stabiliser. Hence X is a topological model of G . The group B is the stabiliser of the fundamental chamber, and X satisfies the assumptions of the previous theorem. Furthermore, the complex-valued cellular chain complex of the *Davis complex* $\Sigma = X/B$ is a projective resolution over $\mathcal{H}(G; B)$. In the sequel q denotes the *thickness* of the building.

As a consequence of Lemma 5.1 we deduce the following statement which was known before [11, Theorem 3.5].

LEMMA 5.3. *The L^2 -Betti numbers of G coincide with the L^2 -Betti numbers $L_q^2 b^i(\Sigma)$ as defined in [11].*

There is the following formula relating the *Poincaré polynomial*

$$\omega(t) = \sum_{w \in W} t^{l(w)} \in \mathbb{Z}[[t]]$$

of the Weyl group $W = N/(B \cap N)$ to the Euler characteristic of G .

THEOREM 5.4 ([11, Corollary 3.4]). *Let q be the thickness of X . The Haar measure ν is normalised so that $\nu(B) = 1$. Then*

$$\chi(G, \nu) = \frac{1}{\omega(q)}.$$

The value $\omega(q)$ can be explicitly computed for the various types of Coxeter systems. We refer to the reader to [6, Section 7.1]. Together with the vanishing results of [12] this can be used for non-trivial computations of L^2 -Betti numbers of algebraic groups over non-Archimedean fields.

5.2. Groups with weakly normal open amenable subgroups. We now turn to a very different class of totally disconnected groups which includes the Neretin group. Here we obtain a vanishing result which generalizes [4, Theorem 1.3].

THEOREM 5.5. *Let $O < G$ be an open amenable subgroup such that for any finite sequence g_1, \dots, g_n of elements in G the intersection of conjugates $O^{g_1} \cap \dots \cap O^{g_n}$ is noncompact. Then all L^2 -Betti numbers of G vanish.*

PROOF. Consider the simplicial complex X whose set of n -simplices is the $(n+1)$ -fold product $G/O \times \dots \times G/O$ with the obvious projections as face maps. There is a natural diagonal G -action on X which makes X into a smooth G -CW-complex. In particular, X is a CW-complex with a cellular G -action. The space X is contractible. The stabiliser of an p -cell σ is an intersection of conjugates of O

$$G_\sigma = O^{g_1} \cap \dots \cap O^{g_n}$$

which is a noncompact amenable subgroup. Let P_* be a $\mathcal{H}(G)$ -resolution by projective right modules of the trivial module \mathbb{C} . Let $C_* = C_*(X) \otimes_{\mathbb{C}} L_\infty^2(G)$ where $C_*(X)$ is the cellular chain complex of X with \mathbb{C} -coefficients. With the obvious diagonal G -action C_* becomes a chain complex of discrete G -modules. By Remark 4.6 C_* thus becomes a chain complex of $\mathcal{H}(G)$ -modules. We can easily identify this module structure via the isomorphisms

$$C_p \cong \bigoplus_{\sigma} \mathbb{C}[G_\sigma \backslash G] \otimes_{\mathbb{C}} L_\infty^2(G) \cong \bigoplus_{\sigma} \mathcal{H}(G) \otimes_{\mathcal{H}(G_\sigma)} L_\infty^2(G)$$

where σ runs through representatives of orbits of p -cells. The right hand side has an obvious left $\mathcal{H}(G)$ -module structure, and the isomorphism between C_p and the right hand side is $\mathcal{H}(G)$ -linear.

Now consider the double complex $D_{**} := P_* \otimes_{\mathcal{H}(G)} C_*$. By well known homological algebra one obtains two spectral sequences converging to the total homology [8, Chapter VII.7]. Since tensoring with a projective module is flat and X is

contractible, the E^1 -term of the first one is

$$E_{p,q}^1 = H_q(P_* \otimes_{\mathcal{H}(G)} C_*) = P_p \otimes_{\mathcal{H}(G)} H_q(C_*) = \begin{cases} P_p \otimes_{\mathcal{H}(G)} L_\infty^2(G) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So the spectral sequence converges to $H_p(G, L_\infty^2(G))$. The second spectral sequence has the E^1 -term

$$E_{p,q}^1 = H_p(P_* \otimes_{\mathcal{H}(G)} C_q) \cong H_p(G, \bigoplus_{\sigma} \mathcal{H}(G) \otimes_{\mathcal{H}(G_\sigma)} L_\infty^2(G)) \cong \bigoplus_{\sigma} H_p(G_\sigma, L_\infty^2(G)).$$

The last isomorphism is the Shapiro isomorphism. Since G_σ is amenable and non-compact, the dimension of every $E_{p,q}^1$ vanishes [22, Theorem 7.10]. Hence also the dimension of $H_*(G, L^2(G))$, i.e. the L^2 -Betti numbers of G vanish. \square

The Neretin group is an interesting totally disconnected group to which we can apply the previous theorem. The required open amenable subgroup is the group O introduced after Theorem 1.1 in [3]. It is shown in *loc. cit.* that the Neretin group is a simple locally compact group without any lattices. Moreover, the Neretin group admits a topological model with cocompact n -skeleton for every $n \geq 0$ [30]. The following corollary solves Problem 1.1 in [22].

COROLLARY 5.6. *The L^2 -Betti numbers of Neretin groups vanish.*

5.3. Low degree Morse inequalities. In this section we provide an upper bound for the first L^2 -Betti number of a compactly generated group.

The correct analogue of the Cayley graph for a compactly generated totally disconnected group G is the Cayley-Abels graph. The Cayley-Abels graph also arises as the 1-skeleton of a suitable topological model of G . Let $K \subset G$ be a compact and open subgroup. Let $S_0 \subset G$ compact so that $S = KS_0K$ is a compact generating set of G . For example, S_0 can be a generating set of a dense subgroup of G . The *Cayley-Abels graph* $\text{Cay}(G, K, S)$ is the quotient of the Cayley graph of (G, S) (edges defined on the left) by the left K -action. So the vertices are cosets Kg and there is an edge (but these are no longer labelled) from Kg to Ksg .

Next we record some familiar properties of the Cayley-Abels graph $\text{Cay}(G, K, S)$ together with its right G -action. First of all, $\text{Cay}(G, K, S)$ has one orbit $K \backslash G$ of vertices. The degree of $\text{Cay}(G, K, S)$ is $K \backslash S$. But there is a difference to the discrete case, where the degree of any vertex is exactly the number of edge-orbits of the action. Here, it is possible that the edge (Ke, Ks) and (Ke, Ksk_0) are translates of each other. The orbit of edges correspond to double cosets $K \backslash S / K$, whose number might not be the degree. The stabiliser of the edge (Ke, Ks) is equal to

$K \cap s^{-1}Ks = K \cap K^s$. Thus, as a G -set, the edges can be identified with

$$\bigsqcup_{KsK \in K \backslash S / K} (K \cap K^s) \backslash G.$$

Thus we obtain an exact sequence

$$\bigoplus_{KsK \in K \backslash S / K} \mathbb{C}[(K \cap K^s) \backslash G] \rightarrow \mathbb{C}[K \backslash G] \rightarrow \mathbb{C} \rightarrow 0$$

of right $\mathcal{H}(G)$ -modules. This can be used to estimate the low-degree L^2 -Betti numbers. Indeed if G is not compact, then $\beta_0^{(2)}(G) = 0$. Together with $[K : K \cap K^s] \cdot \nu(K \cap K^s) = \nu(K)$, we obtain in this case that

$$\beta_1^{(2)}(G, \nu) \leq \frac{1}{\nu(K)} \cdot \left(\sum_{KsK \in K \backslash S / K} [K : K \cap K^s] - 1 \right).$$

This formula is well known in the case where G is discrete, $K = \{1\}$, and ν is just the counting measure. But already when G is discrete and $K \subset G$ is some finite subgroup, we obtain new upper bounds on the first L^2 -Betti number of G . In the non-discrete case, the estimate relates the number of generators and the distortion of a compact-open subgroup K by the generators to the first L^2 -Betti number.

There are analogous lower and upper bounds for combinations of higher L^2 -Betti numbers that arise from the structure of the topological model. Indeed, if G admits a topological model with finite n -skeleton, then the usual Morse inequalities take the form

$$(-1)^n \sum_{i=0}^n (-1)^i \beta_i^{(2)}(G) \leq (-1)^n \sum_{i=0}^n (-1)^i \sum_{U \in \mathcal{F}_i} \mu(U)^{-1},$$

where \mathcal{F}_i is the set of i -cells and U the corresponding stabilizer. In particular, we can use these inequalities easily to derive bounds for low degree L^2 -Betti numbers.

Let us just mention one concrete application concerning the existence of central extensions of lattices.

COROLLARY 5.7. *Let G be a unimodular, totally disconnected group and let $(\Gamma_i)_{i \in \mathbb{N}}$ be a Farber sequence of lattices. If $\beta_2^{(2)}(G, \nu) > 0$, then Γ_i admits a non-trivial central extension for i large enough.*

PROOF. This is a straightforward application of Theorem 1.3. \square

5.4. L^2 -Betti numbers of locally compact groups and their lattices.

In this subsection we provide a short proof of (1.1). Let $C_* \rightarrow \mathbb{C}$ be a projective resolution of the trivial $\mathcal{H}(G)$ -module. Each projective $\mathcal{H}(G)$ -module is a filtered limit of modules of the form $\mathbb{C}[K \backslash G]$ where $K < G$ is a compact open subgroup

(cf. Section 2.3). Since

$$\mathbb{C}[K \backslash G] = \bigoplus_{Ks\Gamma \in K \backslash G/\Gamma} \mathbb{C}[(K^s \cap \Gamma) \backslash \Gamma]$$

as a right $\mathbb{C}[\Gamma]$ -module, $\mathbb{C}[K \backslash G]$ is projective as a right $\mathbb{C}[\Gamma]$ -module. Note that it is a finitely generated projective module if and only if $K \backslash G/\Gamma$ is finite, i.e., if and only if Γ is cocompact. The natural inclusion

$$\begin{aligned} \mathbb{C}[K \backslash G] \otimes_{\mathbb{C}[\Gamma]} \ell^2(\Gamma) &= \bigoplus_{Ks\Gamma \in K \backslash G/\Gamma} \mathbb{C}[(K^s \cap \Gamma) \backslash \Gamma] \otimes_{\mathbb{C}[\Gamma]} \ell^2(\Gamma) \\ &= \bigoplus_{Ks\Gamma \in K \backslash G/\Gamma} \ell^2((K^s \cap \Gamma) \backslash \Gamma) \\ &\subset \ell^2\left(\bigsqcup_{Ks\Gamma \in K \backslash G/\Gamma} (K^s \cap \Gamma) \backslash \Gamma\right) \\ &= \ell^2(K \backslash G) = \mathbb{C}[K \backslash G] \otimes_{\mathcal{H}(G)} L_\infty^2(G) \end{aligned}$$

is a $L(\Gamma)$ -dimension isomorphism since the algebraic sum is rank dense in the ℓ^2 -sum above. The latter follows from the fact that Γ has finite covolume and hence

$$\sum_{Ks\Gamma \in K \backslash G/\Gamma} \dim_{L(\Gamma)} \ell^2((K^s \cap \Gamma) \backslash \Gamma) = \sum_{Ks\Gamma \in K \backslash G/\Gamma} \frac{1}{|K^s \cap \Gamma|} < \infty.$$

Thus, the natural map

$$H_*(C_* \otimes_{\mathbb{C}[\Gamma]} \ell^2(\Gamma)) \rightarrow H_*(C_* \otimes_{\mathcal{H}(G)} L_\infty^2(G))$$

is a $L(\Gamma)$ -dimension isomorphism (one uses exactness of rank completion [32] and the local criterion [26, Theorem 2.4; 32, Theorem 1.2]). Since $\dim_{L(\Gamma)}(M) = \text{covol}_\nu(\Gamma) \cdot \dim_{(L(G), \text{tr})}(M)$ for any right $L(G)$ -module, we obtain the desired equality (1.1).

5.5. The Connes Embedding Problem for Hecke-von Neumann algebras. The Connes Embedding Problem is a major open problem in the theory of von Neumann algebras. It asserts that every finite von Neumann algebra with a separable pre-dual can be embedded into an von Neumann algebraic ultraproduct of the hyperfinite II_1 -factor. We call finite von Neumann algebras for which such an embedding exists *embeddable*. Loosely speaking, a finite von Neumann algebra is embeddable if the joint moments of any finite subset of elements can be approximated by the joint moments of complex matrices.

Typical examples of finite von Neumann algebras are group von Neumann algebras. Then, the embeddability of group von Neumann algebras is related to other famous open problems, such as Gromov's question whether all discrete groups are sofic, see Pestov's survey for more information on these notions [21].

Another source of finite von Neumann algebras are so-called Hecke-von Neumann algebras of Hecke pairs (G, K) . Let G be a locally compact, totally disconnected, second countable, unimodular group with Haar measure μ and let $K \subset G$ be a compact open subgroup. As before, we denote by $L(G) \subset B(L^2(G, \mu))$ the group von Neumann algebra and by $L(G, K) = p_K L(G) p_K$ the Hecke-von Neumann algebra associated with the Hecke pair (G, K) . As we have seen, the Hecke-von Neumann algebra admits a unital faithful and positive normal trace, and thus it is a finite von Neumann algebra. It is natural to ask under which circumstances we are able to give a positive answer to the Connes Embedding Problem. Our main result in this direction is the following theorem.

THEOREM 5.8. *Let G be a locally compact, totally disconnected, second countable, unimodular group and let $K \subset G$ be a compact open subgroup. If G admits a Farber sequence $(\Gamma_i)_{i \in \mathbb{N}}$, then the finite von Neumann algebra $L(G, K)$ is embeddable.*

PROOF. The proof is a side-product of the techniques that were developed in order to prove Theorem 3.4. Indeed, for any $T \in \mathcal{H}(G, K) \subset L(G, K)$, we consider the sequence of matrices $P_{\Gamma_i} \pi_i(T) P_{\Gamma_i} \in B(\ell^2(\bar{X}_{e,i}))$. Note that the operator norm of $P_{\Gamma_i} \pi_i(T) P_{\Gamma_i}$ is bounded independent of $i \in \mathbb{N}$. It remains to show that for any finite list $T_1, \dots, T_k \in \mathcal{H}(G, K)$ the joint moments of $P_{\Gamma_i} \pi_i(T_1) P_{\Gamma_i}, \dots, P_{\Gamma_i} \pi_i(T_k) P_{\Gamma_i}$ converge to the joint moments of T_1, \dots, T_k as the parameter i tends to infinity. However, this is obvious from the constructions in the proof of Theorem 3.4. Now, this implies that

$$\mathcal{H}(G, K) \ni T \mapsto (P_{\Gamma_i} \pi_i(T) P_{\Gamma_i})_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} B(\ell^2(\bar{X}_{e,i}))$$

induces a trace-preserving embedding of $\mathcal{H}(G, K)$ into an von Neumann algebraic ultra-product of matrix algebras. This finishes the proof. \square

Appendix A. Invariant random subgroups

The set Sub_G of closed subgroups of a locally compact group G carries the *Chabauty topology* which is generated by two types of sets, namely

$$\begin{aligned} O_1(C) &= \{H \in \text{Sub}_G \mid H \cap C = \emptyset\}, \quad C \subset G \text{ compact}, \\ O_2(U) &= \{H \in \text{Sub}_G \mid H \cap U \neq \emptyset\}, \quad U \subset G \text{ open}. \end{aligned}$$

Let us now justify the results that we claimed in the introduction. Proposition 1.4 in the introduction is a consequence of the following lemma.

LEMMA A.1. *A sequence (Γ_i) of lattices in G converges as invariant random subgroups to the trivial invariant random subgroup δ_e if and only if for every compact*

open subgroup $K < G$ and for every right coset $C \neq K$ of K the probability that a conjugate of $g\Gamma_i g^{-1}$ meets C tends to zero for $i \rightarrow \infty$.

PROOF. First we show the \Leftarrow -statement. By the Portmanteau theorem weak convergence $\mu_{\Gamma_i} \rightarrow \delta_{\{e\}}$ is equivalent to $\limsup_{i \rightarrow \infty} \mu_{\Gamma_i}(F) \leq \delta_{\{e\}}(F)$ for every closed $F \subset \text{Sub}_G$, and equivalent to $\liminf_{i \rightarrow \infty} \mu_{\Gamma_i}(V) \geq \delta_{\{e\}}(V)$ for every open $V \subset \text{Sub}_G$. Let $K < G$ be a compact open subgroup. It suffices to show that

$$(A.1) \quad \liminf_{i \rightarrow \infty} \mu_{\Gamma_i}(V) \geq \delta_{\{e\}}(V)$$

for the elements V of a subbasis of the Chabauty topology. To show (A.1) we may assume that $\{e\} \in V$. Consider first the case $V = O_2(U)$ for some open $U \subset G$. Then $\{e\} \in V$ means $e \in U$ and hence $\mu_{\Gamma_i}(V) = 1$ for all $i \in \mathbb{N}$. Now let $V = O_1(C)$ for some compact $C \subset G$. Then $\{e\} \in V$ means $e \notin C$. We can find a finite number of cosets $h_k K \neq K$ such that C is contained in their union. By the assumption,

$$\lim_{i \rightarrow \infty} \mu_{\Gamma_i}(O_1(h_k K)) = 1$$

for every k . Hence

$$\lim_{i \rightarrow \infty} \mu_{\Gamma_i}\left(\bigcap_k O_1(h_k K)\right) = 1$$

for the finite intersection. Since $\bigcap_k O_1(h_k K)$ is contained in V we obtain (A.1) in this case.

Next we show the \Rightarrow -direction. Assume that $\mu_{\Gamma_i} \rightarrow \delta_{\{e\}}$ in IRS_G . Let $K < G$ be a compact and open subgroup. For every coset $hK \neq K$ the subset $O_2(hK)$ is closed and open in the Chabauty topology. Let $h \notin K$. From $O_2(hK)$ being closed and weak convergence we deduce that

$$\limsup_{i \rightarrow \infty} \mu_{\Gamma_i}(\{H \mid H \cap hK \neq \emptyset\}) = \limsup_{i \rightarrow \infty} \mu_{\Gamma_i}(O_2(hK)) \leq \delta_{\{e\}}(O_2(hK)) = 0. \quad \square$$

If $A_H = \{H \mid H \cap K \neq \{e\}\}$ is closed in the Chabauty topology, then the Farber condition for the coset $C = K$ in Definition 1.1 would follow from weak convergence $\mu_{\Gamma_i} \rightarrow \delta_{\{e\}}$. Hence in this case the Farber condition is equivalent to $\mu_{\Gamma_i} \rightarrow \delta_{\{e\}}$. We remark that A_H is Chabauty closed if $K \subset G$ is any compact subset and G is a Lie group. This is based on the observation that if (g_i) , $g_i \neq e$, is a bounded sequence in a Lie group G , then there is a sequence (n_i) of integers so that $(g_i^{n_i})$ converges, upon passing to a subsequence, to an element $g \in G \setminus \{1\}$.

On the other side, there are easy examples of sequence of lattices $(\Gamma_i)_{i \in \mathbb{N}}$ with increasing covolumes, which converge to the trivial subgroup as IRS but do not form Farber sequences. Indeed, just take $G = \prod_{n \in \mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$ and $\Gamma_i = \prod_{n=i}^{2i} (\mathbb{Z}/2\mathbb{Z})$.

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